

Prove by mathematical induction: $\sum_{i=1}^n (2i-1)^2 = \frac{n(2n-1)(2n+1)}{3}$ for all integers $n \geq 1$.

SCORE: ____ / 30 PTS

HINT: The algebra in the proof is much easier if you use factoring.

$$\sum_{i=1}^1 (2i-1)^2 = 1^2 = 1 = \frac{1(1)(3)}{3}$$

ASSUME $\sum_{i=1}^k (2i-1)^2 = \frac{k(2k-1)(2k+1)}{3}$ FOR SOME PARTICULAR BUT ARBITRARY INTEGER $k \geq 1$

$$\begin{aligned}\sum_{i=1}^{k+1} (2i-1)^2 &= \sum_{i=1}^k (2i-1)^2 + (2k+1)^2 \\ &= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2 \\ &= (2k+1) \left[\frac{k(2k-1)}{3} + 2k+1 \right]\end{aligned}$$

$$= (2k+1) \left[\frac{2k^2 - k + 6k + 3}{3} \right]$$

$$= \frac{(2k+1)(2k^2 + 5k + 3)}{3}$$

$$= \frac{(2k+1)(k+1)(2k+3)}{3}$$

SO, BY MATHEMATICAL INDUCTION,

$$\sum_{i=1}^n (2i-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

FOR ALL INTEGERS $n \geq 1$

Find rectangular equations corresponding to the parametric equations $x = \frac{t}{2-t}$, $y = \frac{t+3}{t-3}$.

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Write y as a function of x .

$$x(2-t) = t$$

$$2x - tx = t$$

$$2x = t + tx$$

$$2x = t(1+x)$$

$$t = \frac{2x}{1+x}$$

$$y = \frac{\frac{2x}{1+x} + 3}{\frac{2x}{1+x} - 3} \cdot \frac{1+x}{1+x}$$

$$y = \frac{2x + 3(1+x)}{2x - 3(1+x)}$$

$$y = \frac{5x+1}{-x-3} \text{ or } -\frac{5x+1}{x+3}$$

Last year was a good year for the Lee's family business. During January 2012, revenues were \$19,173 and operating costs were \$5,418.

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NOTE: You must show the use of the relevant sequence and/or series formulae to earn full credit for the following problems.

- [a] If revenues each month were 1.5% higher than revenues the previous month, what was the total revenue from January 2012 to August 2012 ?

$$\frac{19173(1.015^8 - 1)}{1.015 - 1} = \$161,682.82$$

- [b] If operating costs each month decreased by a fixed amount (in dollars) from the previous month, and operating costs in August 2012 were \$4,144, what were the total operating costs from January 2012 to December 2012 ?

$$4144 = 5418 + 7d$$

$$d = -182$$

$$\frac{12}{2}(2(5418) + 11(-182)) = \$53,004$$

Consider the sequence defined recursively by $a_n = n^2 - 2a_{n-1}$, $a_1 = 2$.

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Find the first 4 terms of the sequence, and write them as a list.

$$a_2 = 2^2 - 2a_1 = 4 - 4 = 0$$

$$a_3 = 3^2 - 2a_2 = 9 - 0 = 9$$

$$a_4 = 4^2 - 2a_3 = 16 - 18 = -2$$

2, 0, 9, -2

Find the coefficient of $x^{12}y^{46}$ in the expansion of $(7x^3 - 5y^2)^{27}$.

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Your final answer may use $+$, $-$, $\times$ and positive powers. It may **NOT** use $\dots$, $!$, $\div$, negative exponents **NOR** fractions.

It does **NOT** need to be simplified into a single number. Show all work as demonstrated in lecture.

$$\sum_{i=0}^{27} \binom{27}{i} (7x^3)^{27-i} (-5y^2)^i$$
$$= \sum_{i=0}^{27} \binom{27}{i} 7^{27-i} x^{3(27-i)} (-5)^i y^{2i}$$

$$3(27-i) = 12 \text{ AND } 2i = 46$$
$$\text{IF } i = 23$$

$$\binom{27}{23} 7^4 (-5)^{23}$$
$$= \frac{27 \cdot 26 \cdot 25 \cdot 24}{4 \cdot 3 \cdot 2 \cdot 1} \cdot 7^4 (-5)^{23}$$
$$= 27 \cdot 26 \cdot 25 \cdot 7^4 (-5)^{23}$$

Use sigma notation to write the series $\frac{65}{3} - \frac{62}{6} + \frac{59}{12} - \dots + \frac{35}{3072}$. $\leftarrow$ ARITHMETIC $d = -3$
 $\leftarrow$ GEOMETRIC $r = -2$ SCORE: ____ / 20 PTS

Show clearly the algebra used to find the upper limit of summation (ie. not just by counting).

$$\sum_{n=1}^{11} \frac{65-3(n-1)}{3 \cdot (-2)^{n-1}} = \sum_{n=1}^{11} \frac{68-3n}{3 \cdot (-2)^{n-1}}$$

$$\begin{aligned} 68-3n &= 35 \\ \text{IF } n &= 11 \end{aligned}$$

Find parametric equations for the ellipse with $(-1, -5)$ and $(-1, 11)$ as foci, and $(1, 3)$ as one endpoint of the minor axis.

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$$F \cdot (-1, 11)$$

$$C \cdot (-1, 3) \cdot (1, 3)$$

$$F \cdot (-1, -5)$$

$$a^2 = 2^2 + 8^2$$

$$a^2 = 68$$

$$a = 2\sqrt{17}$$

$$x = -1 + 2\cos\theta$$

$$y = 3 + 2\sqrt{17}\sin\theta$$